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The vacuum stress tensor for automorphic fields on some flat space–times

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Abstract. We calculate the vacuum stress–energy tensor for scalar fields having an explicit U(2) symmetry and for neutrinos in three flat space–times whose constant-t hypersurfaces are a 3-torus, Klein bottle and twisted 3-torus. For special values of the parameters, we regain previously calculated values.

1. Introduction

In the preceding paper (Banach and Dowker 1979) we outlined the general theory of fields on space–times with non-trivial fundamental group (automorphic fields) and treated some of the mathematical questions which arose. The approach was in many ways complementary to that of Isham (Isham 1978a, b, Avis and Isham 1978a) who classifies inequivalent vector bundle cross-sections on general space–times as twisted fields. If a space–time has a non-trivial fundamental group then a twisted field generally shows up as an automorphic field on the covering space for some representation of the fundamental group. On the other hand many automorphic fields are in fact gauge related provided one introduces the appropriate connection fields and hence represent the same twisted field. However the automorphic formalism, being a rigid-gauge formalism, avoids the need for additional connections which are not already present in a theory and the emergence of homotopically equivalent automorphic fields is the price one pays for this simplification.

With the exception of a recent preprint by De Witt *et al* (1978) which includes spinor fields, the only calculations done so far concentrate on single real scalar fields where the gauge group is discrete ($\mathbb{Z}_2$) (thus automorphic fields are also twisted fields due to the absence of connections) and Abelian (Isham 1978a, Avis and Isham 1978b, Dowker and Banach 1978). The purpose of this paper is to extend the results to situations involving continuous and non-Abelian groups and to treat spinor fields in the automorphic formalism. We choose three space–times which are closely related factor spaces of infinite Minkowski space but which have significantly different geometrical properties. They are the torus, Klein bottle and twisted torus (defined in § 2) and we compute the Feynman propagator and hence the vacuum averaged stress tensor on each for a multiplet of scalar fields having a U(2) internal symmetry and for neutrinos. The Klein bottle in fact does not support neutrinos at all and we have to mix up left- and

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right-handed neutrinos before we can get a sensible theory. The result is in fact nothing more than a Dirac bispinor in the Weyl representation.

In § 2 we set up the machinery that we need and then set up and solve the group representation problem in a manner sufficiently general to afford immediate extension to arbitrary groups and factor spaces. Sections 3 and 4 deal with the scalars and spinors respectively. An Appendix deals with the general question of transforming automorphic objects into gauges which are not manifestly group invariant: an important topic since manifestly group invariant gauges are frequently not the ones we would prefer to work with. Pertinent examples arise in § 4. This same problem also arises when we restore full gauge freedom to the fields on the covering space requiring the use of more general group representations as hinted at in Banach and Dowker (1979). It will be the subject of a subsequent paper.

2. Basic formulae and the representation problem

We recall that in general we are interested in a non-simply connected space–time M and its universal covering space $\tilde{M}$ with $M \approx \Gamma/\tilde{M}$ where Γ is a discrete group acting on $\tilde{M}$ and $\Gamma \approx \pi_1(M)$. Fields on $\tilde{M}$ project down to $\Gamma/\tilde{M}$ by the general recipe $\phi \to \overset{\leftarrow a}{\phi}$ with

$$\overset{\leftarrow a}{\phi}(x) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} a(\gamma^{-1}) \phi(\gamma x), \tag{1}$$

where $a(\Gamma)$ is a representation of Γ in the gauge group appropriate to the situation in hand. The basic requirement is the invariance of the action under Γ and for a linear field theory this leads to the conclusion that $a(\Gamma)$ commutes with $K(x, y)$ where $K(x, y)$ is the kernel of the action functional. Full details of this can be found in Banach and Dowker (1979). The kernel itself must be group invariant which implies

$$K(\gamma x, y) = K(x, \gamma^{-1} y). \tag{2}$$

We note that this relation embodies a built-in gauge-fixing procedure in that (2) determines the relationship between the gauges used to express $\phi(x)$ and $\phi(\gamma x)$. The gauge-rotated version of this will be treated in the Appendix but (2) is sufficient for normal purposes.

We next turn to the representation problem which consists of finding all admissible $a(\Gamma)$ given Γ and the gauge group G of which $a(\Gamma)$ is to be a discrete subgroup. It turns out that homotopy groups are usually expressed in generator-relation format and so we will review this topic briefly at this point.

A group Γ is said to be generated by $\gamma_1, \gamma_2, \ldots, \gamma_n$ if every $\gamma \in \Gamma$ can be written as a word in the γ_i, by which we mean a finite sequence whose elements are taken from the set of generators and understood to be multiplied as they stand. Thus

$$\gamma = \gamma_{i_1}^{n_1} \gamma_{i_2}^{n_2} \ldots \gamma_{i_k}^{n_k}, \tag{3}$$

where the n_i are positive or negative integers or zero and signify a subword consisting of $|n_i|$ copies of γ_i or its inverse or the identity element respectively. The set of all words in the γ_i is a group X, say, (usually different from Γ) and X is a free group. Now every group is a quotient of a free group on a set of generators by some normal free subgroup; thus $\Gamma = X/R$ and R is generated by some subset of X, Δ say, where $\Delta = \{r_j\}$ (j in some indexing set) and thus Γ is described completely by the pair $(\{\gamma_i\}, \{r_j\})$. Such a description of a group is called a presentation and if both $\{\gamma_i\}$ and $\{r_j\}$ are finite sets, the group is said to be finitely presented.

Now consider a word of the form $\omega = ar_jb$, where a and b are arbitrary subwords and $r_j \in \Delta$ (the r_j are called relations). Since R is normal we have

$$\omega = ar_jb = ab\rho \tag{4}$$

for some $\rho \in R$. Since Γ is the set of left R-cosets we can cancel the ρ on the right since $ab\rho$ and ab determine the same element of Γ. This leads to the substitution laws

$$r_j = e, \tag{5}$$

since whenever an r_j (or combination of them) occurs in a word, the above algorithm enables us simply to replace it by the identity.

Brushing aside technicalities, one might think that refinements of the above mechanism may enable us to express any arbitrary word in some sort of minimal 'standard form'. This is the famous word problem, the proof of the unsolvability of which (even for finitely presented groups) constitutes one of those profound no-go theorems which have shaken up mathematics so much this century.

The solution of the representation problem should now be virtually self-evident. Suppose $\Gamma = (\{\gamma_i\}, \{r_j\})$ and for the moment let us suppose $\{r_j\} = \Delta = \varnothing$ and Γ is free; then let a be any function from $\{\gamma_i\}$ to G. The function a immediately extends to the whole of Γ by

$$a(\gamma) = a(\gamma_{i_1}^{n_1}\gamma_{i_2}^{n_2}\dots\gamma_{i_k}^{n_k}) = [a(\gamma_{i_1})]^{n_1}[a(\gamma_{i_2})]^{n_2}\dots[a(\gamma_{i_k})]^{n_k}. \tag{6}$$

That this should be possible for any group G constitutes the definition of an abstract free group (see e.g. Rotman 1965) and is clearly a homomorphism. The relations are equally easy to deal with. Since γ and $\gamma\rho$ ($\rho \in R$) determine the same group element it is clearly necessary and sufficient that

$$a(r_j) = a(e) = e. \tag{7}$$

Thus to find all admissible $a(\Gamma)$ wc simply look for all solutions of (7) from among arbitrary functions $a : \{\gamma_i\} \to G$.

Let us now apply this to the cases we are interested in.

The space–times we are considering are all of the form $T \otimes M$ where M is a factor space of $\mathbb{R}^3$. The torus is generated by the three isometries

$$\begin{aligned} &A_{\mathrm{T}}: (x, y, z) \to (x+L, y, z),\\ &B: (x, y, z) \to (x, y+M, z),\\ &C: (x, y, z) \to (x, y, z+N). \end{aligned} \tag{8}$$

It is obvious that these three commute so that the relations are

$$A_{\mathrm{T}}B = BA_{\mathrm{T}}, \qquad A_{\mathrm{T}}C = CA_{\mathrm{T}}, \qquad BC = CB. \tag{9}$$

Furthermore, because of commutativity we can write any word in 'standard form'

$$\gamma = (A_{\mathrm{T}})^l(B)^m(C)^n, \qquad l, m, n \in \mathbb{Z}, \tag{10}$$

so that any further relations satisfied by the group are redundant.

The (compactified periodic) Klein bottle differs only in the generator A which becomes

$$A_{\mathrm{KB}}: (x, y, z) \to (x+L, -y, z), \tag{11}$$

which changes the relations to

$$A_{KB}B = B^{-1}A_{KB}, \qquad A_{KB}C = CA_{KB}, \qquad BC = CB. \tag{12}$$

A little manipulation of the first of these shows that any power of A_{KB} can be brought to the other side of any power of B provided we change the sign of the exponent of B, where necessary, hence a standard form like (10) applies.

What, for the purposes of this paper, will be called the twisted torus again differs from the previous two only in its A-generator, which is

$$A_{TT}: (x, y, z) \to (x+L, -y, -z), \tag{13}$$

and the relations are

$$A_{TT}B = B^{-1}A_{TT}, \qquad A_{TT}C = C^{-1}A_{TT}, \qquad BC = CB, \tag{14}$$

whence we can clearly write a standard form again.

The particular group we are interested in is $U(2) \approx U(1) \otimes SU(2)/\mathbb{Z}_2$, and the first thing to do is to find a convenient parametrisation of the group. Not surprisingly perhaps, the most convenient arises via the exponential map

$$U(2) \approx U(1) \otimes SU(2)/\mathbb{Z}_2 \ni g = e^{i\theta} \exp(is\hat{\boldsymbol{\rho}} \cdot \boldsymbol{\sigma}) = e^{i\theta}(\mathbb{I} \cos s + i\hat{\boldsymbol{\rho}} \cdot \boldsymbol{\sigma} \sin s), \tag{15}$$

where $\hat{\boldsymbol{\rho}}$ is a unit vector, σ_i are the usual Pauli matrices and $\mathbb{I}$ is the identity matrix. Substituting this into the relations (9), (12) and (14) (with each generator replaced by its representative) gives us conditions to be satisfied by the parameters of the representatives of the generators of the three factor spaces.

We find for the standard forms of the γ the following:

$$\begin{aligned}
&\text{Torus:} \quad a(A_T^l B^m C^n) = \exp i(l\theta_A + m\theta_B + n\theta_C) \\
&\qquad\qquad \times [\mathbb{I} \cos(ls_A + ms_B + ns_C) + i\hat{\boldsymbol{\rho}} \cdot \boldsymbol{\sigma} \sin(ls_A + ms_B + ns_C)] \\
&\text{Klein bottle:} \quad a(A_{KB}^l B^m C^n) = (\pm)^m\, e^{i(l\theta_A + n\theta_C)}[\mathbb{I} \cos(ls_A + ns_C) + i\hat{\boldsymbol{\rho}} \cdot \boldsymbol{\sigma} \sin(ls_A + ns_C)] \\
&\text{Twisted torus:} \quad a(A_{TT}^l B^m C^n) = (\pm)^m (\pm)^n\, e^{il\theta_A}[\mathbb{I} \cos ls_A + i\hat{\boldsymbol{\rho}} \cdot \boldsymbol{\sigma} \sin ls_A],
\end{aligned} \tag{16}$$

where the θ_k and s_k are arbitrary real numbers and $\hat{\boldsymbol{\rho}}$ is an arbitrary unit vector. Note that in each case the representation is actually Abelian.

Finally we recall some geometrical facts about our spaces. The torus is a homogeneous space, being $[\mathbb{Z}/\mathbb{R}]^3$, and as a coset space inherits its orientability and parallelisability from that of $\mathbb{R}^3$. The other two are non-homogeneous. The Klein bottle is not orientable as is practically obvious and it is not parallelisable either, for consider two non-zero everywhere-independent vector fields on $\mathbb{R}^3$, X and Y which are invariant under A_{KB}; i.e.

$$A_{KB}^* X = X, \qquad A_{KB}^* Y = Y. \tag{17}$$

They define a plane in the tangent space at each point. Add a third independent vector at some point to form a basis for vector fields there and consider its behaviour under A_{KB}^*. The (Euclidean) angle between the vector and plane is $\alpha \neq 0$, say, and after application of A_{KB}^* becomes $-\alpha$; thus the intermediate value theorem prevents any continuous everywhere-independent extension to a global parallelisation. This lack of a parallelisation (or spinor structure, see Geroch 1968, 1970) presents certain obstacles to the construction of spinor fields as will be elaborated in § 4.

By contrast, the twisted torus *is* parallelisable (hence orientable). Here is a specific parallelisation:

$$\begin{aligned} X_0 &= \partial_t, \qquad X_1 = \partial_x, \\ X_2 &= \cos(\pi x/L)\partial_y + \sin(\pi x/L)\partial_z, \\ X_3 &= -\sin(\pi x/L)\partial_y + \cos(\pi x/L)\partial_z. \end{aligned} \tag{18}$$

We will use this structure explicitly to construct manifestly group-invariant neutrino fields on the twisted torus. This space is in many ways the most interesting of the three we consider since it has all the global geometrical properties that we would like but differs non-trivially from the homogeneous torus. It is also not something of a special case like the Klein bottle.

The spaces we consider are not the only possibilities for $\Gamma/\mathbb{R}^3$ of course. The complete classification can be found in Wolf (1967) and some remarks about other cases in De Witt *et al* (1978). Our concern is not so much the physical relevance of the results (which is slight), but the exposition of general methods in a tractable context (hence the detailed presentation). The restriction to the three cases here is adequate from this point of view.

3. The scalar case

We consider a pair of complex massless scalars with the Lagrangian

$$\mathscr{L} = \tfrac{1}{2}(\nabla_\mu \phi^a)^\dagger (\nabla^\mu \phi_a) - (\xi/2)\phi_a^\dagger \phi^a R, \qquad a = 1, 2, \tag{19}$$

where R is the curvature scalar, the a 'metric' which is implied is just the identity and the covariant derivatives are just the partials since ϕ is a scalar. The case $\xi = \frac{1}{6}$ gives the conformally coupled theory (see Callan *et al* 1970) in lowest order and $\xi = 0$ is the usual minimal coupling. This theory has an obvious rigid U(2) gauge symmetry.

From this we can find the energy–momentum tensor in the usual manner and for its vacuum expectation we can write it in terms of the Feynman propagator

$$\mathscr{D}^a_{\mathrm{F}b}(x, x') = \mathrm{i}\langle 0|T(\phi^a(x)\phi_b^\dagger(x'))|0\rangle \tag{20}$$

($|0_{\mathrm{in}}\rangle = |0_{\mathrm{out}}\rangle = |0\rangle$ because $T \otimes M$ is stationary hence (20) is correctly normalised) as (see De Witt 1975, Dowker and Critchley 1976, Schwinger 1951, Dowker and Banach 1978)

$$\begin{aligned} \langle T_{\mu\nu}\rangle = -2\mathrm{i}\ \mathrm{Re} \lim_{x'\to x} \mathrm{tr}[&(1-2\xi)\nabla_\mu \nabla'_\nu + g_{\mu\nu}(2\xi - \tfrac{1}{2})\nabla_\rho \nabla'^\rho - \xi(\nabla_\mu\nabla_\nu + \nabla'_\mu \nabla'_\nu) \\ &+ \xi g_{\mu\nu}(\nabla_\rho \nabla^\rho + \nabla'_\rho \nabla'^\rho) - \xi R_{\mu\nu} + \tfrac{1}{2}\xi R g_{\mu\nu}]\mathscr{D}^a_{\mathrm{F}b}(x, x'). \end{aligned} \tag{21}$$

Strictly speaking, the point split object in (21) is not a geometric object and we would have to remedy this by incorporating parallel propagators as in Dowker and Critchley (1976), but the standard regularisation scheme for image sum type flat spaces is the dropping of the direct term which always gives the infinite Minkowski space constant; and this method is quite insensitive to these minor indiscretions. This can be verified directly, the affinity being integrable, due to the flatness of the spaces making the path integral (see e.g. Boulware and Deser 1967)

$$g^{\mu}_{\nu'} = \int \mathscr{D}[x(\tau)]\left\{\exp \int_{x'}^{x} ds(\tau)^{\alpha}\Gamma^{\mu}_{\alpha\nu}(s(\tau))\right\} \tag{22}$$

equal to just the geodesic contribution, and thus explicitly computable.

Having repeated all of these familiar things, the actual calculations are straightforward. The Feynman propagator is given by

$$\mathscr{D}^{a}_{Fb}(x, x') = \frac{-i}{4\pi^2} \sum_{l,m,n=-\infty}^{+\infty} \frac{e^{i(\boldsymbol{l}\cdot\boldsymbol{\theta})}}{[(x^{\mu} - A^{\mu}_{l}x'^{\mu} - l^{\mu}L^{\mu})\eta_{\mu\nu}(x^{\nu} - A^{\nu}_{l}x'^{\nu} - l^{\nu}L^{\nu}) - i\epsilon]}$$
$$\times \begin{bmatrix} \cos(\boldsymbol{l}\cdot\boldsymbol{s}) + i\gamma\sin(\boldsymbol{l}\cdot\boldsymbol{s}), & (i\alpha+\beta)\sin(\boldsymbol{l}\cdot\boldsymbol{s}) \\ (i\alpha-\beta)\sin(\boldsymbol{l}\cdot\boldsymbol{s}), & \cos(\boldsymbol{l}\cdot\boldsymbol{s}) - i\gamma\sin(\boldsymbol{l}\cdot\boldsymbol{s}) \end{bmatrix}, \tag{23}$$

where $\hat{\boldsymbol{\rho}} = (\alpha, \beta, \gamma)$, $\alpha^2+\beta^2+\gamma^2 = 1$, $\boldsymbol{l}\cdot\boldsymbol{s}$ and $\boldsymbol{l}\cdot\boldsymbol{\theta}$ take the values permitted by (16), $l^{\mu}L^{\mu} = (0, lL, mM, nN)$ and no contraction is implied (similarly for $A^{\mu}_{l}x'^{\mu}$):

$$A^{\mu}_{l} = +1 \qquad \text{for the torus}$$
$$\begin{cases} -1 & \text{for } \mu = 2 \text{ and } l \text{ odd} \\ +1 & \text{otherwise} \end{cases} \qquad \text{for the Klein bottle}$$
$$\begin{cases} -1 & \text{for } \mu = 2 \text{ or } 3 \text{ and } l \text{ odd} \\ +1 & \text{otherwise} \end{cases} \qquad \text{for the twisted torus.}$$

It is sometimes said that one can 'regularise' the propagator by dropping the $\boldsymbol{l} = 0$ term to get a regularised $\mathscr{D}_F(x, x')$. This is incorrect. The basic reason is that to define an acceptable function on a factor space we take an image sum over a group. 'Translation invariance' in the group then ensures that certain periodicity conditions are satisfied by the sum and hence that one fundamental domain is much the same as any other. Removing part of the sum destroys the periodicity conditions and so invalidates the remainder as a function on the factor space *unless* the removed part itself satisfies the correct periodicity conditions by some miracle. This certainty does not apply to the $\boldsymbol{l} = 0$ terms above—if it did there would be no need for the image sum at all. In the coincidence limit, most of the position dependence drops out, the $\boldsymbol{l} = 0$ term gives a constant (albeit an infinite one) and the result turns out to be acceptable.

We can now calculate $\langle T_{\mu\nu}\rangle$ quite generally from (21). The fifth and sixth terms obviously do not contribute and the fourth one does not either since it is only non-zero on the term that regularisation throws away. The rest can be written as (cf. Dowker and Banach 1978)

$$\langle T_{\mu\nu}\rangle = \frac{-1}{2\pi^2}\sum_{\boldsymbol{l}}{}' 4\cos(\boldsymbol{l}\cdot\boldsymbol{\theta})\cos(\boldsymbol{l}\cdot\boldsymbol{s})\left\{(1-2\xi)A_{\nu l}\left(\frac{\eta_{\mu\nu}}{H(\boldsymbol{l})^2} + \frac{4Z_{\mu l}Z_{\nu l}}{H(\boldsymbol{l})^3}\right)\right.$$
$$+ (2\xi - \tfrac{1}{2})\eta_{\mu\nu}\left[\eta^{\alpha\beta}A_{\beta l}\left(\frac{\eta_{\alpha\beta}}{H(\boldsymbol{l})^2} + \frac{4Z_{\alpha l}Z_{\beta l}}{H(\boldsymbol{l})^3}\right)\right]$$
$$\left. + \xi(1 + A_{\mu l}A_{\nu l})\left(\frac{\eta_{\mu\nu}}{H(\boldsymbol{l})^2} + \frac{4Z_{\mu l}Z_{\nu l}}{H(\boldsymbol{l})^3}\right)\right\}, \tag{24}$$

where

$$Z_{\mu l} = \lim_{x'\to x} (x_{\mu} - A_{\mu l}x'_{\mu} - l_{\mu}L_{\mu}) \tag{25}$$

and

$$H(\boldsymbol{l}) = -Z_{\mu l} Z^{\mu}_{l}. \tag{26}$$

Formula (24) in fact is general enough to apply to any of the spaces $T \otimes \Gamma/\mathbb{R}^3$ provided we interpret the group representation factors and the $A_{\mu l}$ correctly. After that it is just a question of collecting terms. For the cases of interest to us we find:

Torus:

$$\langle T_{00}\rangle_{\mathrm{T}} = 4\left(\frac{-1}{2\pi^2}\right) \sum_{l}{}' \frac{\Phi_{\mathrm{T}}(\boldsymbol{l})}{H_{\mathrm{T}}(\boldsymbol{l})^2}, \tag{27}$$

$$\langle T_{ii}\rangle_{\mathrm{T}} = 4\left(\frac{1}{2\pi^2}\right) \sum_{l}{}' \frac{\Phi_{\mathrm{T}}(\boldsymbol{l})}{H_{\mathrm{T}}(\boldsymbol{l})^2}\left[1 - \frac{4l_i^2 L_i^2}{H_{\mathrm{T}}(\boldsymbol{l})}\right], \tag{28}$$

$$\langle T_{0i}\rangle_{\mathrm{T}} = 0, \qquad \langle T_{ij}\rangle_{\mathrm{T}} = 4\left(\frac{1}{2\pi^2}\right) \sum_{l}{}' \frac{\Psi_{\mathrm{T}ij}(\boldsymbol{l})}{H_{\mathrm{T}}(\boldsymbol{l})^3}[4 l_i L_i l_j L_j], \tag{29}$$

where

$$H_{\mathrm{T}}(\boldsymbol{l}) = l^2 L^2 + m^2 M^2 + n^2 N^2 \tag{30}$$

and $\Phi_{\mathrm{T}}(\boldsymbol{l})$ and $\Psi_{\mathrm{T}ij}(\boldsymbol{l})$ are just the parts of $\cos(\boldsymbol{l}\cdot\boldsymbol{s})\cos(\boldsymbol{l}\cdot\boldsymbol{\theta})$ having the correct evenness–oddness properties in l, m, n to give non-vanishing contributions to $\langle T_{\mu\nu}\rangle$.

$\Phi_{\mathrm{T}}(\boldsymbol{l})$ is completely even, while $\Psi_{\mathrm{T}ij}(\boldsymbol{l})$ is odd in l_i and l_j and even in l_k. Both contain the four possible terms consisting of products of six factors of the form

$$\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\left(l_i\begin{Bmatrix}\theta_i\\ s_i\end{Bmatrix}\right),$$

each θ_i and s_i appearing once in each term.

Klein bottle:

$$\langle T_{00}\rangle_{\mathrm{KB}} = 4\left(\frac{-1}{2\pi^2}\right) \sum_{l}{}' \Phi_{\mathrm{KB}}(\boldsymbol{l})\left\{\frac{1}{H^2_{\mathrm{KB1}}(\boldsymbol{l})} + \frac{2(6\xi-1)}{H^2_{\mathrm{KB2}}(\boldsymbol{l})} + \frac{4(4\xi-1)\{\frac{1}{2}[1-(-1)^l]l^2L^2 + n^2N^2\}}{H^3_{\mathrm{KB2}}(\boldsymbol{l})}\right\},$$

$$\langle T_{11}\rangle_{\mathrm{KB}} = 4\left(\frac{1}{2\pi^2}\right) \sum_{l}{}' \Phi_{\mathrm{KB}}(\boldsymbol{l})\left\{\frac{1}{H^2_{\mathrm{KB1}}(\boldsymbol{l})} + \frac{2(6\xi-1)}{H^2_{\mathrm{KB2}}(\boldsymbol{l})} - \frac{4\{\frac{1}{2}[1-(-1)^{l+1}]\}l^2L^2}{H^3_{\mathrm{KB1}}(\boldsymbol{l})} - \frac{16\xi\{\frac{1}{2}[1-(-1)^l]\}l^2L^2 + 4(4\xi-1)n^2N^2}{H^3_{\mathrm{KB2}}(\boldsymbol{l})}\right\},$$

$$\langle T_{22}\rangle_{\mathrm{KB}} = 4\left(\frac{1}{2\pi^2}\right) \sum_{l}{}' \Phi_{\mathrm{KB}}(\boldsymbol{l})\left\{\frac{1}{H^2_{\mathrm{KB1}}(\boldsymbol{l})} - \frac{4m^2M^2}{H^3_{\mathrm{KB1}}(\boldsymbol{l})}\right\},$$

$$\langle T_{33}\rangle_{\mathrm{KB}} = 4\left(\frac{1}{2\pi^2}\right) \sum_{l}{}' \Phi_{\mathrm{KB}}(\boldsymbol{l})\left\{\frac{1}{H^2_{\mathrm{KB1}}(\boldsymbol{l})} + \frac{2(6\xi-1)}{H^2_{\mathrm{KB2}}(\boldsymbol{l})} - \frac{4n^2N^2}{H^3_{\mathrm{KB1}}(\boldsymbol{l})} - \frac{4(4\xi-1)\{\frac{1}{2}[1-(-1)^l]\}l^2L^2 + 16\xi n^2N^2}{H^3_{\mathrm{KB2}}(\boldsymbol{l})}\right\},$$

$$\langle T_{0i}\rangle_{\mathrm{KB}} = 0 = \langle T_{12}\rangle_{\mathrm{KB}} = \langle T_{23}\rangle_{\mathrm{KB}},$$

$$\langle T_{13}\rangle_{\rm KB}=4\left(\frac{1}{2\pi^2}\right)\sum_l{}'\,(\pm)^m[\cos(l\theta_{\rm A})\cos(n\theta_C)\sin(ls_{\rm A})\sin(ns_C)$$
$$+\sin\,(l\theta_{\rm A})\sin(n\theta_C)\cos(ls_{\rm A})\cos(ns_C)]$$
$$\times\frac{4lLnN}{[\![l^2L^2+\{[1-(-1)^l]y-mM\}^2+n^2N^2]\!]^3},\tag{31}$$

where

$$H_{\rm KB1}(l)=\begin{cases}l^2L^2+m^2M^2+n^2N^2 & l\text{ even}\\ \infty & l\text{ odd}\end{cases},\tag{32}$$

$$H_{\rm KB2}(l)=\begin{cases}l^2L^2+(2y-mM)^2+n^2N^2 & l\text{ odd}\\ \infty & l\text{ even}\end{cases},\tag{33}$$

and

$$\Phi_{\rm KB}(l)=(\pm)^m[\cos(l\theta_{\rm A})\cos(n\theta_C)\cos(ls_{\rm A})\cos(ns_C)$$
$$+\sin(l\theta_{\rm A})\sin(n\theta_C)\sin(ls_{\rm A})\sin(ns_C)].\tag{34}$$

Twisted torus:

$$\langle T_{00}\rangle_{\rm TT}=4\left(\frac{-1}{2\pi^2}\right)\sum_l{}'\,\Phi_{\rm TT}(l)\left\{\frac{1}{H^2_{\rm TT1}(l)}+\frac{(8\xi-1)}{H^2_{\rm TT2}(l)}-\frac{4(4\xi-1)\{\frac12[1-(-1)^l]\}l^2L^2}{H^3_{\rm TT2}(l)}\right\},$$

$$\langle T_{11}\rangle_{\rm TT}=4\left(\frac{1}{2\pi^2}\right)\sum_l{}'\,\Phi_{\rm TT}(l)\left\{\frac{1}{H^2_{\rm TT1}(l)}+\frac{(8\xi-1)}{H^2_{\rm TT2}(l)}-\frac{4\{\frac12[1-(-1)^{l+1}]\}l^2L^2}{H^3_{\rm TT1}(l)}\right.$$
$$\left.\frac{-16\xi\{\frac12[1-(-1)^l]\}l^2L^2}{H^3_{\rm TT2}(l)}\right\},$$

$$\langle T_{22}\rangle_{\rm TT}=4\left(\frac{1}{2\pi^2}\right)\sum_l{}'\,\Phi_{\rm TT}(l)\left\{\frac{1}{H^2_{\rm TT1}(l)}-\frac{(4\xi-1)}{H^2_{\rm TT2}(l)}-\frac{4m^2M^2}{H^3_{\rm TT1}(l)}+\frac{(4\xi-1)[4(2z-nN)^2]}{H^3_{\rm TT2}(l)}\right\},$$

$$\langle T_{33}\rangle_{\rm TT}=4\left(\frac{1}{2\pi^2}\right)\sum_l{}'\,\Phi_{\rm TT}(l)\left\{\frac{1}{H^2_{\rm TT1}(l)}-\frac{(4\xi-1)}{H^3_{\rm TT2}(l)}-\frac{4n^2N^2}{H^3_{\rm TT1}(l)}+\frac{(4\xi-1)[4(2y-mM)^2]}{H^3_{\rm TT2}(l)}\right\},$$

$$\langle T_{0i}\rangle_{\rm TT}=0=\langle T_{12}\rangle_{\rm TT}=\langle T_{13}\rangle_{\rm TT},$$

$$\langle T_{23}\rangle_{\rm TT}=4\left(\frac{1}{2\pi^2}\right)\sum_l{}'\,\Phi_{\rm TT}(l)\,\frac{4(1-4\xi)(2y-mM)(2z-nN)}{H^3_{\rm TT2}(l)},\tag{35}$$

where

$$H_{\rm TT1}(l)=\begin{cases}l^2L^2+m^2M^2+n^2N^2 & l\text{ even}\\ \infty & l\text{ odd}\end{cases},\tag{36}$$

$$H_{\rm TT2}(l)=\begin{cases}l^2L^2+(2y-mM)^2+(2z-nN)^2 & l\text{ odd}\\ \infty & l\text{ even}\end{cases}\tag{37}$$

and

$$\Phi_{\rm TT}(l)=(\pm)^m(\pm)^n\cos l\theta_A\cos ls_A.\tag{38}$$

If we take the limits $M, N\to\infty$ we get the results for the infinite slab, infinite Möbius strip and infinite twisted slab. For these cases, the summations can be reduced to Fourier series and with the help of Gradshteyn and Ryzhik (1965 § 1.44), can be expressed eventually in closed form. The resulting expressions are in general so long as to be virtually useless and so we do not list them here. For numerical approximation, the series expressions converge rapidly enough. Dowker and Banach (1978) give a detailed examination of some simpler cases to which the above results reduce if we set $s = 0$.

For ease of comparison, we postpone further discussion until we have listed the spinor results.

4. The spinor case

In a curved space, the left-handed neutrino Lagrangian can be written

$$\mathscr{L} = (\mathrm{i}/2)(\det L)L^{\mu}_{a}(\psi^{\dagger}\sigma^{a}\nabla_{\mu}\psi - (\nabla_{\mu}\psi^{\dagger})\sigma^{a}\psi), \tag{39}$$

with $\sigma^{a} = (\mathbb{I}, -\sigma_x, -\sigma_y, -\sigma_z)$, the usual Pauli matrices; L^{μ}_{a} is a set of tetrads, i.e. a solution of

$$g_{\mu\nu} = L^{a}_{\mu}L^{b}_{\nu}\eta_{ab} \qquad \text{or} \qquad \eta_{ab} = L^{\mu}_{a}L^{\nu}_{b}g_{\mu\nu} \tag{40}$$

with tetrad indices raised and lowered by η_{ab} and Greek indices by the usual g. ψ is a two-component spinor and $\psi^{\dagger}$ its Hermitian conjugate and the covariant derivative is given by

$$\nabla_{\mu}\psi = (\partial_{\mu} + \mathrm{i}B_{\mu})\psi, \qquad \nabla_{\mu}\psi^{\dagger} = \partial_{\mu}\psi^{\dagger} - \mathrm{i}\psi^{\dagger}B_{\mu}, \tag{41}$$

$$B_{\mu} = \tfrac{1}{4}B_{\mu ab}\sigma^{ab}, \qquad \sigma^{ab} = (\mathrm{i}/2)[\sigma^{a}, \sigma^{b}], \tag{42}$$

$$B_{\mu ab} = \Gamma^{\alpha}_{\mu\beta}L_{\alpha a}L^{\beta}_{b} + L_{\alpha b}(\partial_{\mu}L^{\alpha}_{a}). \tag{43}$$

Clearly the only gauge freedom left in (39) for use in automorphic projections is the usual U(1), and so the representations are just given by (16) with the SU(2) factors omitted.

Again we can write the vacuum expectation of the stress–energy tensor in terms of the Feynman propagator

$$S_{\mathrm{F}}(x, x') = -\mathrm{i}\langle 0|T(\psi(x)\psi^{\dagger}(x'))|0\rangle \tag{44}$$

as

$$\langle T_{ab}\rangle = \tfrac{1}{2}\lim_{x'\to x} \operatorname{tr} \sigma_{(a}[\nabla_{b)}S_{\mathrm{F}}(x, x') - S_{\mathrm{F}}(x, x')\nabla'_{b)}], \tag{45}$$

where

$$\nabla_{a} = (\det L)L^{\mu}_{a}\nabla_{\mu}, \tag{46}$$

and we have expressed $\langle T_{\mu\nu}\rangle$ in the tetrad basis. From a formal point of view, it is much better to work in the tetrad basis throughout (cf. Brill and Cohen 1966, Lichnerowicz 1964) since to define a sensible spinor field on the factor space, the tetrad has to be group invariant, i.e.

$$\gamma^{*}L_{a} = L_{a}, \qquad \forall\gamma\in\Gamma, \tag{47}$$

and it follows that everything else is also invariant when expressed in this basis.

For the torus all of the above is the veritable sledgehammer cracking a nut. We take the L^a_μ parallel to the coordinate axes which gives us the flat-space Dirac equation

$$\mathrm{i}\sigma^a \partial_a S_{\mathrm{F}}(x, x') = \delta(x, x') \tag{48}$$

solved by

$$S_{\mathrm{F}}(x, x') = -\mathrm{i}\tau^a \partial_a \mathscr{D}_{\mathrm{F}}(x, x'), \tag{49}$$

with

$$\tau^a = (\mathbb{1}, \sigma_x, \sigma_y, \sigma_z), \tag{50}$$

$$\partial_\mu \partial^\mu \mathscr{D}_{\mathrm{F}}(x, x') = \delta(x, x'), \tag{51}$$

and

$$\mathscr{D}_{\mathrm{F}}(x, x') = -\mathrm{i}[4\pi^2((t-t')^2 - (\boldsymbol{r}-\boldsymbol{r}')^2 - \mathrm{i}\epsilon)]^{-1} \tag{52}$$

being the Minkowski scalar propagator, essentially the $\boldsymbol{l} = 0$ term of (23). Since everything is manifestly Γ-invariant we may as well work with the $\boldsymbol{l} = 0$ term and make the automorphic projection as the last step before the coincidence limit is taken. We find

$$S_{\mathrm{F}}(x, x') = \frac{1}{2\pi^2} \frac{1}{[(t-t')^2 - (\boldsymbol{r}-\boldsymbol{r}')^2 - \mathrm{i}\epsilon]} \begin{bmatrix} (t-t') - (z-z'); & -(x-x') + \mathrm{i}(y-y') \\ -(x-x') - \mathrm{i}(y-y'); & (t-t') + (z-z') \end{bmatrix}, \tag{53}$$

and hence

$$\langle T_{00} \rangle_{\mathrm{T}} = \frac{1}{\pi^2} \sum_{\boldsymbol{l}}{}' \frac{\cos(l\theta_A)\cos(m\theta_B)\cos(n\theta_C)}{H_{\mathrm{T}}(\boldsymbol{l})^2},$$

$$\langle T_{ii} \rangle_{\mathrm{T}} = \frac{1}{\pi^2} \sum_{\boldsymbol{l}}{}' \frac{\cos(l\theta_A)\cos(m\theta_B)\cos(n\theta_C)}{H_{\mathrm{T}}(\boldsymbol{l})^2} \left[\frac{4 l_i^2 L_i^2}{H_{\mathrm{T}}(\boldsymbol{l})} - 1 \right],$$

$$\langle T_{0i} \rangle_{\mathrm{T}} = 0,$$

$$\langle T_{ij} \rangle_{\mathrm{T}} = -\frac{1}{\pi^2} \sum_{\boldsymbol{l}}{}' \frac{\sin(l_i\theta_i)\sin(l_j\theta_j)\cos(l_k\theta_k)}{H_{\mathrm{T}}(\boldsymbol{l})^3} 4 l_i L_i l_j L_j, \tag{54}$$

with $H_{\mathrm{T}}(\boldsymbol{l})$ given by (30) again. Naturally the implied long calculation above is circumvented by putting (49) directly into (45) (similarly to the scalar case) and obtaining (54) more quickly, but it is useful to write out (53) in readiness for considering the twisted torus which we now do.

The spinor structure used for the torus, parallel to the coordinate axes, is unsuitable for the twisted torus; it does not satisfy (47). At another level, the propagator (53) does not satisfy (2) and so cannot be used in a naive image summation. What we require is a spinor structure like (18) on Minkowski space, which does satisfy (47). Putting this into the general formulae we find the explicit Dirac equation

$$\mathrm{i}\left[\partial_t - \sigma_x\left(\partial_x - \frac{\mathrm{i}\pi}{2L}\sigma_x\right) - \sigma_y\left(\cos\left(\frac{\pi x}{L}\right)\partial_y + \sin\left(\frac{\pi x}{L}\right)\partial_z\right) - \sigma_z\left(-\sin\left(\frac{\pi x}{L}\right)\partial_y + \cos\left(\frac{\pi x}{L}\right)\partial_z\right)\right] \times S_{\mathrm{F}}(x, x') = \delta(x, x'). \tag{55}$$

Now Minkowski space is topologically trivial so that all spinor structures are related by SO(1, 3) gauge transformations and these translate into SL(2, $\mathbb{C}$) gauge rotations on the spinors themselves (see Isham 1978b for a full discussion of these points as well as the topological pros and cons of the usual old wives' tales about spinors); thus consider the gauge rotated version of (48):

$$\mathrm{i}[S(x)\sigma^a\partial_a\delta(x,x')S^{-1}(x')][S(x')S_F(x',x'')S^{-1}(x'')] = S(x)\delta(x,x'')S^{-1}(x''). \tag{56}$$

If $S(x)$ is such as to change the differential operator in (48) to the one in (55), the gauge-rotated $S_F(x,x')$ will automatically be the Feynman propagator of (55) as well. The $S(x)$ we need is given by (see Goldstein 1950, p 115)

$$S(x) = \begin{bmatrix} \cos(\pi x/2L) & \mathrm{i}\sin(\pi x/2L) \\ \mathrm{i}\sin(\pi x/2L) & \cos(\pi x/2L) \end{bmatrix}, \tag{57}$$

yielding the propagator

$$S_F(x,x') = \frac{1}{2\pi^2}\frac{1}{[T^2-X^2-Y^2-Z^2-\mathrm{i}\epsilon]^2} \times \begin{bmatrix} T_{C\ominus}-\mathrm{i}X_{S\ominus}+Y_{S\oplus}-Z_{C\ominus}; & \mathrm{i}T_{S\ominus}-X_{C\ominus}+\mathrm{i}Y_{C\oplus}+\mathrm{i}Z_{S\oplus} \\ \mathrm{i}T_{S\ominus}-X_{C\ominus}-\mathrm{i}Y_{C\ominus}-\mathrm{i}Z_{S\oplus}; & T_{C\ominus}-\mathrm{i}X_{S\ominus}-Y_{S\oplus}+Z_{C\oplus} \end{bmatrix}, \tag{58}$$

where

$$T=(t-t'), \qquad X=(x-x'), \qquad Y=(y-y'), \qquad Z=(z-z'),$$
$$C\oplus = \cos[\pi(x\pm x')/2L], \qquad S\oplus = \sin[\pi(x\pm x')/2L]. \tag{59}$$

It is clear that (58) now satisfies (2) and we can proceed with impunity. Actually, if we calculate with the explicitly group-invariant formula (45) we can again leave the image sum until the last step before the coincidence limit. Rotating back to the Cartesian basis will then give us $\langle T_{\mu\nu}\rangle$.

All of this involves considerable labour so we avail ourselves of the mechanism explained in the Appendix. Essentially, this involves applying a gauge transformation to the projected S_F to bring it into the usual flat space gauge and then working with Minkowski differential operators. Expressed in terms of the propagator (53), the twisted torus propagator now takes the form

$$S_F(x,x') = \frac{1}{2\pi^2}\sum_l \frac{\mathrm{e}^{\mathrm{i}l\theta}(\pm)^m(\pm)^n}{\{(t-t')^2-(x-x'-lL)^2-[y-(-1)^l y'-mM]^2-[z-(-1)^l z'-nN]^2-\mathrm{i}\epsilon\}^2}$$
$$\times\begin{bmatrix} (t-t')-[z-(-1)^l z'-nN]; & -(x-x'-lL)+\mathrm{i}[y-(-1)^l y'-mM] \\ -(x-x'-lL)-\mathrm{i}[y-(-1)^l y'-mM]; & (t-t')+[z-(-1)^l z'-nN] \end{bmatrix}$$
$$\times\left\{\begin{bmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}\right\}^l. \tag{60}$$

The factors $[\mathrm{i}\sigma_x]^l$ play the part of the $S(\gamma)\equiv S(\gamma x)S^{-1}(x)$ of the Appendix (A15). After some algebra we find

$$\langle T_{00}\rangle_{\mathrm{TT}} = \frac{1}{\pi^2}\sum_l{}' \frac{\cos(2l\theta)(\pm)^m(\pm)^n(-1)^l}{H_{\mathrm{TT1}}(l)^2},$$

$$\langle T_{11}\rangle_{\mathrm{TT}} = \frac{1}{\pi^2}\sum_l{}' \frac{\cos(2l\theta)(\pm)^m(\pm)^n(-1)^l}{H_{\mathrm{TT1}}(l)^2}\left[\frac{4(2l)^2L^2}{H_{\mathrm{TT1}}(l)}-1\right],$$

$$\langle T_{22}\rangle_{\rm TT}=\langle T_{33}\rangle_{\rm TT}=\frac{1}{\pi^2}\sum_l{}' \frac{\cos(2l\theta)(\pm)^m(\pm)^n(-1)^l}{H_{\rm TT1}(l)^2}\left[\frac{4l_i^2L_i^2}{H_{\rm TT1}(l)}-1\right],$$

$$\langle T_{01}\rangle_{\rm TT}=\frac{2}{\pi^2}\sum_l \frac{\sin((2l+1)\theta)(\pm)^m(\pm)^n(-1)^l(2l+1)^2L^2}{H_{\rm TT2}(l)^3},$$

$$\langle T_{02}\rangle_{\rm TT}=\langle T_{03}\rangle=0,$$

$$\langle T_{12}\rangle_{\rm TT}=\frac{-2}{\pi^2}\sum_l \frac{\cos((2l+1)\theta)(\pm)^m(\pm)^n(-1)^l(2l+1)L(2z-nN)}{H_{\rm TT2}(l)^3},$$

$$\langle T_{13}\rangle_{\rm TT}=\frac{2}{\pi^2}\sum_l \frac{\cos((2l+1)\theta)(\pm)^m(\pm)^n(-1)^l(2l+1)L(2y-mM)}{H_{\rm TT2}(l)^3}$$

$$\langle T_{23}\rangle_{\rm TT}=0, \tag{61}$$

with

$$H_{\rm TT1}(l)=(2l)^2L^2+m^2M^2+n^2N^2, \tag{62}$$

$$H_{\rm TT2}(l)=(2l+1)^2L^2+(2y-mM)^2+(2z-nN)^2. \tag{63}$$

Note that the sums involving $H_{\rm TT2}(l)$ are unprimed; in (61) l is used to label (even and odd) subsets of integers and is no longer a group element label; thus the '$l=0$' term automatically disappears in the odd case.

There remains the Klein bottle. As proved in § 2, no spinor structure exists here and so there is no possibility of constructing a manifestly group-invariant theory of neutrinos at all.

At the propagator level, this shows up in the failure of (2) to hold with the group operations of the Klein bottle and any gauge rotated version of (53); specifically

$$S_{\rm F}(A_{\rm KB}x, x')\neq S_{\rm F}(x, A_{\rm KB}^{-1}x'). \tag{64}$$

We might care to examine where the equality breaks down; it is simply a sign change in the $(y-y')$ term. If we could, by a similarity transformation, interchange the off-diagonal terms in (53) leaving the others alone, then we could set up a transformation law as in the Appendix (A7) and hence construct an acceptable theory. It is not difficult to see that this is impossible. What however *is* possible is

$$S_{\rm F}^{\rm L}(A_{\rm KB}x, x')=[{\rm i}\sigma_y]S_{\rm F}^{\rm R}(x, A_{\rm KB}^{-1}x')[-{\rm i}\sigma_y], \tag{65}$$

$$S_{\rm F}^{\rm R}(A_{\rm KB}x, x')=[-{\rm i}\sigma_y]S_{\rm F}^{\rm L}(x, A_{\rm KB}^{-1}x')[{\rm i}\sigma_y], \tag{66}$$

where $S_{\rm F}^{\rm L}$ is the left-handed propagator (53) and $S_{\rm F}^{\rm R}$ is the right-handed neutrino propagator which differs from (53) merely by a sign change in the $(x-x')$, $(y-y')$ and $(z-z')$ terms. This should not surprise us. The $A_{\rm KB}$ generator of the Klein bottle is essentially a parity transformation. Neutrinos have no self-contained covariance under parity; instead, left- and right-handed neutrinos are interchanged (see e.g. Bade and Jehle 1953) and thus the mixing up of left and right neutrinos on a Klein bottle is to be expected.

We can rewrite (65), (66) as

$$S_{\rm F}^{\rm W}(A_{\rm KB}x, x')=[{\rm i}\gamma_y]S_{\rm F}^{\rm W}(x, A_{\rm KB}^{-1}x')[-{\rm i}\gamma_y], \tag{67}$$

with

$$S_{\mathrm{F}}^{\mathrm{W}}(x, x') = \begin{bmatrix} 0 & S_{\mathrm{F}}^{\mathrm{L}}(x, x') \\ S_{\mathrm{F}}^{\mathrm{R}}(x, x') & 0 \end{bmatrix} \tag{68}$$

and

$$\gamma_0 = \begin{bmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix}, \qquad \gamma_i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}, \tag{69}$$

and we recognise the propagator for a massless Dirac bispinor in the Weyl representation which is simply a pair of left- and right-handed neutrinos; and $\mathrm{i}\gamma_y = \gamma_0\gamma_x\gamma_z$ is the parity operator.

By (A15) the Klein bottle propagator is now (l labels the A_{KB} power)

$$S_{\mathrm{F(KB)}}^{\mathrm{W}}(x, x') = \sum_{\gamma\in\Gamma} \begin{bmatrix} 0 & S_{\mathrm{F}}^{\mathrm{L}}(x, \gamma x') \\ S_{\mathrm{F}}^{\mathrm{R}}(x, \gamma x') & 0 \end{bmatrix} \left\{ \begin{bmatrix} 0 & \mathrm{i}\sigma_y \\ -\mathrm{i}\sigma_y & 0 \end{bmatrix} \right\}^{l}. \tag{70}$$

Note that for odd l, the non-zero part of the propagator is thrown into the diagonal sub-blocks of $S_{\mathrm{F}}^{\mathrm{W}}$. Taking traces with the Weyl representation γ_μ, equation (69), as required by (45), reveals that the odd l terms make no contribution to $\langle T_{\mu\nu}\rangle$ and we immediately regain the result of De Witt *et al* (1978) that $\langle T_{\mu\nu}\rangle$ on the Klein bottle for a bispinor is given by the expression for a torus of dimensions $(2L, M, N)$ with the $a(\gamma)$ phase factors modified to the Klein bottle ones.

This completes the presentation of the results.

5. Discussion

It is time to clothe the rather naked formulae of the previous two sections with some commentary. Firstly we want to comment on their general self-consistency. For the scalar case, reducing the SU(2) factors to the unit element, and the U(1) factors to ± 1, gives us the previously obtained results for standard and twisted real scalar fields contained in Dowker and Banach (1978) and De Witt *et al* (1978), apart from the factor of four which is a degrees of freedom adjustment. For the spinors, reduction of the U(1) factors to ± 1 gives us the results for standard and twisted spin connections, discussed so comprehensively in Isham (1978b). The classification theory presented there shows that spinor connections are classified by the same cohomology group as real scalars so one might anticipate the ± 1 factors on those grounds; but we have much more direct confirmation of this, namely the results of the twisted-torus neutrino calculation. If we drop the odd-l image terms from the calculation, we arrive at a torus of dimensions $(2L, M, N)$ but this time equipped with a rotating spin connection. We see that apart from trivial adjustments to the U(1) factors, the resulting $\langle T_{\mu\nu}\rangle$ is just that for the ordinary torus with the spinor field satisfying an anti-periodicity condition

$$\psi(A_{\mathrm{T}}^{l}x) = (-1)^{l}\psi(x). \tag{71}$$

The vanishing of the off-diagonal components of $\langle T_{\mu\nu}\rangle$ is now caused by the restricted nature of these U(1) factors for the twisted torus. All of this agrees precisely with the simple twisted spinor calculation given in De Witt *et al* (1978).

We can just as easily consider twisted spin connections on the twisted torus itself. In this case we would need to rotate the connection through 3π rather than π upon application of A_{TT}. This would cause the $[i\sigma_x]$ factor in (60) to be changed to $[i\sigma_x]^3 = [-i\sigma_x]$ and hence would make no difference to the even-l terms and make the odd-l terms change sign. Thus the diagonal part of $\langle T_{\mu\nu}\rangle$ would stay the same while the off-diagonal part would change sign. Another way to see the same thing is to retain only the $l = 3k$ terms in the twisted torus calculation, giving a twisted torus of dimensions $(3L, M, N)$ and a spin connection which rotates through 3π along the x direction. Rescaling then gives the required result.

By the Isham result, there are no spinor connections (apart from ones also twisted in the y and z directions which, as is fairly obvious, are all taken care of by the U(1) factors set to ± 1) which are not equivalent to one of these. This is a striking illustration of the 2-1 nature of the $SL(2, \mathbb{C}) \to SO(1, 3)$ covering; rotating frames through 2π sends ψ to $-\psi$ and a rotation of 4π is needed to regain the original ψ.

Note the contrast between the $\langle T_{ij}\rangle$ terms for the spinor and scalar fields on the twisted torus. The non-zero ones in the one case are precisely the zero ones in the other and *vice versa*. It would be hard to find firmer confirmation of the comments in Dowker and Critchley (1976) that destroying global Poincaré invariance in a space–time removes any covariance necessity for a tensor constructed from a field theory to be proportional to $\eta_{\mu\nu}$. At the same time note the non-vanishing $\langle T_{01}\rangle$ for the twisted torus spinors. Its presence is in disagreement with the stated vanishing of $\langle T_{0i}\rangle$ for static space–times in Dowker and Kennedy (1978), who do not consider any twisting up of a field, the basic cause of the non-zero $\langle T_{01}\rangle$. Admittedly one can make $\langle T_{01}\rangle$ vanish by suitable choice of θ, either 0 or π, which are just the values you need to give to θ to obtain the untwisted and twisted spinors of Isham's classification, and this leads us to ask what is the physical significance (if any) of the various $a(\gamma)$ factors that we can derive.

As has been stated before (Dowker and Banach 1978), the biggest clue we have comes from the Aharonov–Bohm effect (see e.g. Schulman 1971). There, in a one-particle formalism, the Hamiltonian fails to be given simply by its expression as a differential operator and specific boundary conditions (our $a(\gamma)$ representation factors) have to be included before the differential operator defines an essentially self-adjoint operator on a Hilbert space. In a many-body formalism like field theory, the same situation arises, as the most cursory examination of the Fock–Cook construction (Cook 1953, Emch 1972) reveals. Thus we are tempted to identify the $a(\gamma)$ with fluxes of external gauge potentials which pass through the 'holes' in our space–time. For the U(1) factors we find the usual interpretation of $a(\textit{generator})$ being proportional to the electromagnetic flux, exactly as in the Aharonov–Bohm effect, while for the SU(2) factors, one might set up a similar scheme interpreted as, perhaps, isospin.

All of this works quite nicely for tori where we can 'see' the holes and picture the flux going through them (although we carefully avoid the philosophical minefield that opens up before us when we start talking about an electromagnetic field situated outside the universe), but for the non-homogeneous cases we find some problems. Certain of the $a(\textit{generator})$ are restricted to a discrete set of values and one is naturally tempted to speculate about the 'topological quantisation of the external potential' but a little thought shows that nothing in a geometrical picture can set the value of a flux that is capable of being drawn on paper to one of a discrete rather than continuous set of values. Then again, even such a simple space as the two-dimensional Klein bottle is a one-sided surface and hence the inside and outside of one of its holes are the same; so where exactly is the flux supposed to go? In these cases, the discrete possibilities for $a(\gamma)$ clearly signal the presence of homotopically inequivalent bundle cross-sections.

The cases where there are continuous families of $a(\gamma)$ signal equally clearly the presence of some degree of homotopic redundancy. The fact that we can move smoothly from one representation to another means that if we gauge the rigid degrees of freedom, gauge transformations will render equivalent these representations. For example on $T \otimes S^1$ (an example considered in Isham 1978a) there are no inequivalent complex scalar fields but *we* find a one-parameter family (depending on θ_A) by suitably restricting the results for the torus in § 3. The introduction of an electromagnetic external gauge field A_μ into the problem trivialises all of these θ_A-dependent U(1) bundles and we recover Isham's result (Isham, private communication).

Nevertheless, it should not be thought that all continuous families of representations should be discarded in favour of just one of their members. Consider the case of the neutrinos in § 4. The U(1) factors are all trivialisable (for the torus say) but they are only U(1)-trivialisable. Performing the U(1) trivialisation would throw away the twisted spinor case which is *not* SL(2, ℂ)-trivialisable and hence counts as an inequivalent spinor result. Thus we find that the continuous families of representations often interpolate between genuinely inequivalent bundle cross-sections. We have to examine the cohomology results to ensure we do not throw away important cases.

Returning briefly to $T \otimes S^1$ we remark that there is one possibility for complex scalars not considered in the above, and that is

$$\phi(x+L) = \phi^*(x), \tag{72}$$

where * is charge conjugation. This is inequivalent to the cases considered since it corresponds to enlarging the gauge group from SO(2) to O(2). There is no reason to exclude this as it is a *bona fide* symmetry of the Lagrangian and remarks about charge conservation violation in this and related situations can be found in Kiskis (1978). The reason we mention it here is that an analogous mechanism is at work in the Klein bottle spinor calculation where the enlargement of the gauge group is from SL(2, ℂ) to SL(2, ℂ) ⊗ SL(2, ℂ)* where SL(2, ℂ)* denotes the conjugate action of SL(2, ℂ), appropriate to right-handed neutrinos. This gauge group covers the enlargement of SO(1, 3) to include the parity transformation which lies outside the identity component of O(1, 3) and hence is not in SO(1, 3). One consequence of this is the impossibility (as we saw) of satisfying (2), and so we may generally presume that when we deal with gauge transformations outside the identity component the same applies, and we have to weaken (2) to at least

$$K(\gamma x, y) = S(\gamma) K(x, \gamma^{-1} y) S^{-1}(\gamma), \tag{73}$$

and more generally to (A7).

Finally, we comment on the SU(2) representations themselves. As noted in § 2, they are actually Abelian and thus cannot be irreducible. In fact, in the notation of (15)

$$U(t, \hat{\boldsymbol{n}}) U(s, \hat{\boldsymbol{p}}) U(-t, \hat{\boldsymbol{n}}) = U[s, \cos 2t\hat{\boldsymbol{p}} + \sin 2t(\hat{\boldsymbol{p}} \times \hat{\boldsymbol{n}}) + (1 - \cos 2t)(\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}}], \tag{74}$$

and so we can bring $\hat{\boldsymbol{p}}$ to the z direction by a gauge transformation, diagonalising $a(\Gamma)$ which now explicitly represents two independent complex fields. The independence of the physical results of $\hat{\boldsymbol{p}}$ is a consequence of this gauge freedom in the problem and the flexibility of interpretation that goes with it.

To conclude, we have shown how to deal with fields on multiply connected spaces in a fairly general manner. In the rigid gauge framework of automorphic field theory, the fields are either members of continuous families (a reflection of the infinite nature of the discrete group) or of discrete ones, and we have demonstrated a limited degree of success in interpreting the former by analogy with the Aharonov–Bohm effect. The latter case does not seem to yield such an interpretation easily.

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Finally we would like to thank Chris Isham and Bryce De Witt and their co-workers for keeping us posted about their work prior to publication.

Appendix

In this Appendix we cast some of the general theory of projection operators corresponding to discrete groups into a gauge rotated form. Let us start with a free field Lagrangian

$$\mathscr{L} = \phi^{\dagger}(x)K(x, y)\phi(y), \tag{A1}$$

where the dagger represents a suitable adjoint operator and $K(x, y)$ is the appropriate linear operator. As in Banach and Dowker (1979), the group invariance of the action leads to the group invariance of $\mathscr{L}$ and thus

$$\mathscr{L} = \phi^{\dagger}(x)K(x, y)\phi(y) = \phi^{\dagger}(x)a^{-1}(\gamma)K(\gamma x, \gamma y)a(\gamma)\phi(y). \tag{A2}$$

Requiring that the action of K and Γ commute leads to

$$K(\gamma x, y) = K(x, \gamma^{-1}y), \qquad \forall\gamma \in \Gamma, \tag{A3}$$

and hence, from (A2), we conclude that $a(\Gamma)$ and $K(x, y)$ commute. Thus we see that the permitted gauge group for projections, $a(\Gamma)$, is one which actually leaves K invariant and as such deals with degrees of freedom for which gauge connection fields have *not* been introduced into K (if they had, local gauge transformations would render equivalent many automorphic fields and we would be in the domain of Isham's twisted fields); it is thus a rigid gauge group.

For degrees of freedom that *are* gauged in K, we note (as in § 2) that (A3) or its equivalent

$$K(\gamma x, \gamma y) = K(x, y) \tag{A4}$$

gives a gauge fixing procedure for fields at x and at γx. Since different gauges must be regarded as equivalent, we need a way of pairing up the gauges at x and γx. Since the Lagrangian is gauge covariant, it does not yield a gauge fixing rule itself—we must impose some condition *a priori*, and the commuting of K and Γ (and hence (A3)) is the most natural one.

Now suppose we are interested in working in a different gauge for which the fields are transformed as

$$\phi(x) \to \phi_S(x) = S(x)\phi(x), \tag{A5}$$

where $S(x)$ is some general position-dependent gauge transformation which does not encroach upon the $a(\Gamma)$ degrees of gauge freedom. $K(x, y)$ then becomes

$$K(x, y) \to K_S(x, y) = S(x)K(x, y)S^{-1}(y), \tag{A6}$$

and (A3) becomes

$$K_S(\gamma x, y)=S(\gamma x)K(\gamma x, y)S^{-1}(y)=S(\gamma x)K(x, \gamma^{-1}y)S^{-1}(y)$$
$$=S(\gamma x)S^{-1}(x)K_S(x, \gamma^{-1}y)S(\gamma^{-1}y)S^{-1}(y). \quad (A7)$$

Noting that $a(\Gamma)$ commutes with $S(x)$, we find for the projected fields

$$\overset{\leftarrow a}{\phi}(x)=\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} a(\gamma^{-1})\phi(\gamma x)\rightarrow \overset{\leftarrow a}{\phi_S}(x)=S(x)\overset{\leftarrow a}{\phi}(x)$$
$$=\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} a(\gamma^{-1})S(x)S^{-1}(\gamma x)\phi_S(x), \quad (A8)$$

and for the projected linear operators

$$\overset{\leftrightarrow a}{K}(x, y)=\frac{1}{|\Gamma|^2}\sum_{\gamma,\gamma'\in\Gamma} a(\gamma^{-1})K(\gamma x, \gamma' y)a(\gamma')\rightarrow \overset{\leftrightarrow a}{K_S}(x, y)=S(x)\overset{\leftrightarrow a}{K}(x, y)S^{-1}(y)$$
$$=\frac{1}{|\Gamma|^2}\sum_{\gamma,\gamma'\in\Gamma} a(\gamma^{-1})S(x)K(x, \gamma^{-1}\gamma' y)S^{-1}(y)a(\gamma')$$
$$=\frac{1}{|\Gamma|^2}\sum_{\gamma,\gamma'\in\Gamma} S(x)S^{-1}(x)K_S(x, \gamma^{-1}\gamma' y)S(\gamma^{-1}\gamma' y)a(\gamma^{-1}\gamma')$$
$$=\frac{1}{|\Gamma|}\sum_{\tilde\gamma\in\Gamma} K_S(x, \tilde\gamma y)S(\tilde\gamma y)S^{-1}(y)a(\tilde\gamma). \quad (A9)$$

If it now happens (as it does in § 4) that we can arrange that the factors $S(\gamma x)S^{-1}(x)$ are in fact independent of x, i.e.

$$S(\gamma x)S^{-1}(x)\equiv S(\gamma), \quad (A10)$$

then

$$S(\gamma)=S(\gamma x)S^{-1}(x)=S[\gamma(\gamma^{-1}x)]S^{-1}(\gamma^{-1}x)=S(x)S^{-1}(\gamma^{-1}x); \quad (A11)$$

hence

$$S(\gamma^{-1})=S(\gamma^{-1}x)S^{-1}(x)=S^{-1}(\gamma) \quad (A12)$$

and

$$S(\gamma_1)S(\gamma_2)=S(\gamma_1 x)S^{-1}(x)S(x)S^{-1}(\gamma_2^{-1}x)=S(\gamma_1 x)S^{-1}(\gamma_2^{-1}x)$$
$$=S(\gamma_1\gamma_2 x')S^{-1}(x')=S(\gamma_1\gamma_2), \quad (A13)$$

so that $S(\Gamma)$ is a representation.

Equations (A8) and (A9) now become

$$\overset{\leftarrow a}{\phi_S}(x)=\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} a(\gamma^{-1})S(\gamma^{-1})\phi_S(\gamma x), \quad (A14)$$

$$\overset{\leftrightarrow a}{K_S}(x, y)=\overset{\rightarrow a}{K_S}(x, y)=\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} K_S(x, \gamma y)S(\gamma)a(\gamma), \quad (A15)$$

which look very much like the unrotated forms given in Banach and Dowker (1979), the difference being the gauge factors $S(\gamma)$ which no longer commute with $K_S(x, y)$. The significance of the factors $S(\gamma)$ is of course perfectly clear. They simply compensate for the different quantities of gauge rotation we choose at the points x and γx. If $S(\Gamma) \equiv \mathbb{I}$ then $S(x)$ is an explicitly group-invariant gauge transformation on the covering space and in that case the theory using the rotated or unrotated form is the same.

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